False Theorems and Fake Proofs

Or, Ten Ways to Show that $1 = 2$

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We present several elementary results in the burgeoning field of Alternative Facts (AF, or AFC if the Axiom of Choice is assumed). These "theorems" are all utterly false, but their proofs sound surprisingly convincing and break down in unexpectedly subtle ways. The purpose of these proofs is not only to provide humor to those who read them with a twinkle in their eye, but also to open up the machinery of mathematical arguments to those who don't. The point is to teach rigor through humor: laughter is the reward for understanding.

Contents

1 Warming Up

1.1 Right for the Wrong Reason

Theorem 1.1 (Long Division)**.** *$3 \times 9 = 27$.*

Proof.

$$3 \cdot 9 = 3\sqrt{81} = 3 \begin{array}{|r} \multicolumn{1}{r}{27} \\ \hline 81 \\ \multicolumn{1}{r}{6} \\ \hline \multicolumn{1}{r}{21} \\ \multicolumn{1}{r}{21} \\ \hline \multicolumn{1}{r}{0} \end{array}$$

■

Theorem 1.2 (Cancellation)**.** *$\frac{16}{64} = \frac{1}{4}$.*

Proof.

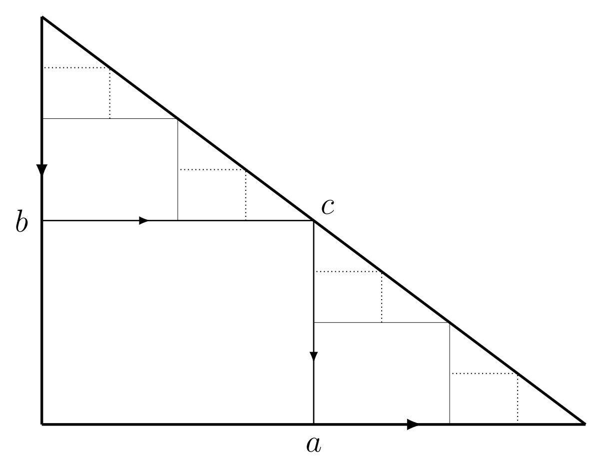
$$\frac{16}{64} = \frac{1\cancel{6}}{\cancel{6}4} = \frac{1}{4}. \tag{1.1}$$

There are three other anomalous two-digit fractions: $\frac{19}{95} = \frac{1}{5}$, $\frac{26}{65} = \frac{2}{5}$, and $\frac{49}{98} = \frac{4}{8}$. ■

1.2 The Drunk Pythagorean Theorem

Theorem 1.3 ("Pythagoras")**.** *In a right triangle with sides a, b, c as shown, $c = a + b$.*

Proof. We bisect a and b, draw perpendiculars to the middle of c, and observe that the path indicated by arrows has the same length $a + b$ as that of the two legs of the triangle. We repeat the procedure, bisecting the legs of the similar right triangles generated by our first cuts. Following a similar jagged path brings us closer to c, but still preserves the total length of the path, $a + b$. This procedure may be repeated indefinitely; passing to the limit, we see that a path of length $a + b$ converges to a line of length c; hence $c = a + b$. ■



Corollary 1.4. $3 + 4 = 5$.

Proof (1). Take the Pythagorean triple $(3, 4, 5)$; then $3 + 4 = 5$ by Thm. 1.3. ■

2 Basic Number Theory

In this section, we invoke the spirit of Georg Cantor to prove the following:

1. 1 is the largest natural number.
2. All $n \in \mathbb{N}$ are equal.
3. All $n \in \mathbb{N}$ are equal to zero.
4. All $n \in \mathbb{N}$ are a whole lot less than a million, but all $n \in \mathbb{N}$ are very large.
5. Zero is a very large number, and $1 = 0$.

2.1 The Largest Number

Theorem 2.1. *1 is the largest natural number.*

Proof. Suppose, to the contrary, that some $n > 1$ is the largest number. Then n^2 must be strictly less than n (since n is the largest, and $n \neq 1$), and therefore $n^2 - n = n(n-1) < 0$. But this is impossible: both factors, n and $n-1$, are positive, so their product $n(n-1)$ must also be positive. Therefore our assumption is false, and 1 is the largest natural number. ■

2.2 Total Equality

Theorem 2.2 (Marx)**.** *All $n \in \mathbb{N}$ are equal.*

Proof. Let n denote the larger of two natural numbers: $n := \max\{a, b\}$. We will show by induction that for all $n \in \mathbb{N}$, $a = b$. Indeed, when $n = 0$, the requirement that $a, b \in \mathbb{N}$ forces $a = b = 0$. Now suppose for some $k \in \mathbb{N}$ that $a = b$. Then,

$$\max\{a, b\} = k + 1 \implies \max\{a - 1,\ b - 1\} = k, \tag{2.1}$$

so by the inductive hypothesis $a - 1 = b - 1$, and therefore $a = b$. Hence $a = b$ even for $n = k + 1$; this completes the inductive step, and we conclude that any two natural numbers are equal to their maximum. Equivalently, all natural numbers are equal. ■

Theorem 2.3 (Nihilism)**.** *All $n \in \mathbb{N}$ are equal to zero.*

Proof. We proceed by strong induction on n. Certainly for $n = 0$, zero is equal to zero. Next, we assume for *all* $k \leq n$ that $k = 0$. In particular, $1 = 0$. Then $k + 1 = 0 + 0 = 0$, which completes the inductive step and proves that $n = 0$ for all $n \in \mathbb{N}$. ■

2.3 Very Large Numbers

Theorem 2.4 (Law of Large Numbers)**.** *All $n \in \mathbb{N}$ are a whole lot less than a million, but all $n \in \mathbb{N}$ are very large.*

Proof. We first show that all $n \in \mathbb{N}$ are a whole lot less than a million, and then that all $n \in \mathbb{N}$ are very large, both by induction on n, starting with $n = 1$. Surely 0 is a whole lot less than a million; and if some k is a whole lot less than a million, then so is $k + 1$. Now by Thm. 2.1, 1 is the largest natural number and therefore very large. And given some very large $k \in \mathbb{N}$, $k + 1$ must also be very large. So all $n \in \mathbb{N}$ are very large numbers a whole lot less than a million. ■

Corollary 2.5. *Zero is a very large number, and* $1 = 0$*.*

Proof (2). Since all $n \in \mathbb{N}$ equal zero (Thm. 2.3) and all $n \in \mathbb{N}$ are very large (Thm. 2.4), zero is very large. Moreover, ten billion is a whole lot less than a million (in particular, less than 1, the largest number), and by Thm. 2.2 it must equal 1, and both must equal zero. ■

3 "Real" Analysis

In this section, we begin our quest to write down as many proofs that $1 = 2$ as possible. Throughout, *Proof (n)* denotes the n^{th} proof that $1 = 2$.

Remark 3.1. The statement that $1 = 2$ is equivalent (by induction) to the statement that $n = 0$ for any $n \in \mathbb{N} \setminus \{0\}$. In light of this, Cors. 1.4 and 2.5 constitute our first and second proofs of this fact; many more follow.

The classic statement and proof of our main theorem follows:

Theorem 3.2 (Fundamental "Theorem" of Numbers)**.** $1 = 2$*.*

Proof (3). Let $a = b$ be nonzero quantities: then, $a = b$ implies that

$$a^2 = ab \implies a^2 - b^2 = ab - b^2 \implies (a-b)(a+b) = b(a-b) \implies a + b = b, \tag{3.1}$$

and since $a = b$, we get $a + b = b + b = 2b = b \implies 2 = 1$. ■

3.1 Complex Exponentials

Theorem 3.3 (Square Roots)**.** $1 = -1$*.*

Proof (4). $1 = \sqrt{(-1)(-1)} = \sqrt{-1} \cdot \sqrt{-1} = i \cdot i = i^2 = -1$. ■

Theorem 3.4 (Transcendentals)**.** *For all $x \in \mathbb{R}$, $e^x = 1$.*

Proof. $e^x = \exp(2\pi i \cdot \frac{x}{2\pi i}) = [\exp(2\pi i)]^{x/2\pi i} = 1^{x/2\pi i} = (1^{1/2\pi i})^x = 1^x = 1.$ ∎

Proof (5). Take $x = \ln(2)$ in Thm. 3.4; then $e^{\ln(2)} = 2 = 1$. ∎

Theorem 3.5. *For all $n \in \mathbb{N}$ and for all nonzero $a \in \mathbb{R}$, $a^n = 1$.*

Proof. We proceed by strong induction on n: the case $n = 0$ gives $a^0 = 1$, which is true by definition. Assuming that $a^k = 1$ for all $k \leq n \in \mathbb{N}$, we have

$$a^{k+1} = a^{2k-(k-1)} = \frac{a^k \cdot a^k}{a^{k-1}} = \frac{1 \cdot 1}{1} = 1, \tag{3.2}$$

where $a^k = a^{k-1} = 1$ by the inductive hypothesis. ∎

Proof (6). Take $a = \sqrt{2}$ and $n = 2$ in Thm. 3.5; then $\sqrt{2}^2 = 2 = 1$. ∎

3.2 Methods of Calculus

We next apply differential and integral calculus to muster two more proofs of Thm. 3.2:

Proof (7). Observe first that

$$\begin{aligned} 1^2 &= 1 \cdot 1 = 1; \\ 2^2 &= 2 \cdot 2 = 2 + 2; \\ 3^2 &= 3 \cdot 3 = 3 + 3 + 3, \text{ etc.} \end{aligned} \tag{3.3}$$

By analogy, $x^2 = x \cdot x = \overbrace{x + \cdots + x}^{x \text{ times}}$. Taking the derivative of both expressions, we get

$$\frac{\mathrm{d}}{\mathrm{d}x}(x^2) = 2x; \qquad \frac{\mathrm{d}}{\mathrm{d}x}(x + \cdots + x) = \underbrace{1 + \cdots + 1}_{x} = x. \tag{3.4}$$

Hence $2x = x \implies 1 = 2$. ∎

Proof (8). Observe that $\frac{1}{x} = 2\frac{1}{2x}$. We integrate both expressions $\mathrm{d}x$, using the fact that $[\ln(2x)]' = 2\frac{1}{2x}$ (from the chain rule) to make a substitution in the second integral:

$$\int \frac{1}{x}\,\mathrm{d}x = \ln|x|; \qquad \int 2\frac{1}{2x}\,\mathrm{d}x = \ln|2x|. \tag{3.5}$$

Hence $\ln|x| = \ln|2x|$, and taking $x = 1$ we have $\ln(1) = \ln(2) \implies 1 = 2$. ∎

4 Advanced Dark Wizardry

In this section, we continue our quest to find proofs that $1 = 2$ using more advanced methods. We then give a two more profoundly baffling results before closing on a humorous note.

4.1 Series and Transforms

Proof (9). Recall that a series for $\ln(2)$ can be obtained from the Taylor series for $\ln(1+x)$:

$$\ln(1+x) = \sum_{i=1}^{\infty} \frac{(-1)^{n-1}}{n} x^n \implies \ln(2) = \sum_{i=1}^{\infty} \frac{(-1)^{n-1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots \tag{4.1}$$

We rearrange the terms of this series; indeed, it is not too hard to check that

$$\begin{aligned} \ln(2) &= \sum_{i=1}^{\infty} \left[\left(\frac{1}{2n-1} - \frac{1}{2(2n-1)} \right) - \frac{1}{4n} \right] = \\ &= \left(1 - \frac{1}{2}\right) - \frac{1}{4} + \left(\frac{1}{3} - \frac{1}{6}\right) - \frac{1}{8} + \left(\frac{1}{5} - \frac{1}{10}\right) - \frac{1}{12} + \cdots = \\ &= \frac{1}{2} - \frac{1}{4} + \frac{1}{6} - \frac{1}{8} + \frac{1}{10} - \frac{1}{12} + \cdots = \frac{1}{2}\ln(2). \end{aligned} \tag{4.2}$$

Hence $\ln(2) = \frac{1}{2}\ln(2)$, and so $1 = \frac{1}{2} \implies 1 = 2$. ■

Proof (10). Consider the function $f\colon \mathbb{R} \to \mathbb{R}$ given by $f(x) = 1$. Its Fourier transform $\hat{f}$, the Dirac delta, is zero almost everywhere, so $\hat{\hat{f}} \equiv 0$ identically. And because $\hat{\hat{f}} = f$ by the inversion theorem, we immediately have $1 = 0$. ■

4.2 Two Last Results

Theorem 4.1 (Fundamental "Theorem" of Calculus)**.** *For all $a \in \mathbb{R}$ and any integrable function $f\colon [0,a] \to \mathbb{R}$,*

$$I := \int_0^a f(x)\ \mathrm{d}x = 0. \tag{4.3}$$

Proof. We make a substitution, beloved by physicists: $u := \sin\left(\frac{\pi x}{a}\right) \implies \mathrm{d}u = \frac{\pi}{a}\cos\left(\frac{\pi x}{a}\right)$:

$$\cos\left(\frac{\pi x}{a}\right) = \sqrt{1 - \sin^2\left(\frac{\pi x}{a}\right)} = \sqrt{1-u^2} \implies \mathrm{d}x = \frac{a}{\pi}\frac{\mathrm{d}u}{\sqrt{1-u^2}}. \tag{4.4}$$

Observe that $x = 0 \implies u = 0$, and $x = a \implies u = 0$, so that our integral becomes

$$I = \int_0^a f(x)\ \mathrm{d}x = \frac{\pi}{a}\int_0^0 \frac{f\left(\frac{a}{\pi}\sin^{-1} u\right)}{\sqrt{1-u^2}}\ \mathrm{d}u = 0. \tag{4.5}$$

■

Theorem 4.2. $\mathbb{R}$ *is countable.*

Proof (I). Since $\mathbb{Q}$ is dense in $\mathbb{R}$, between any two real numbers $a \neq b$ can be found a rational number q. This defines a bijection $(a, b) \mapsto q$ from pairs of real numbers to rational numbers, so $\mathbb{R}$ has the same cardinality as $\mathbb{Q}$, and therefore $\mathbb{R}$ is countable. ■

Proof (II). We will explicitly construct a bijection $\mathbb{N} \to \mathbb{R}$. Choose any $a_1 \in \mathbb{N}$ and any $b_1 \in \mathbb{R}$, and define $f_{a_1} \colon \mathbb{N} \to \mathbb{R}$ by $f_{a_1}(a_1) = b_1$. After sending $a_1 \mapsto b_1$, excise them from their respective sets and consider $\mathbb{N} \setminus \{a_1\}$ and $\mathbb{R} \setminus \{b_1\}$. Choose $(a_2, b_2) \in \mathbb{N} \times \mathbb{R} \setminus (a_1, b_1)$ and again define $f_{a_2} \colon \mathbb{N} \to \mathbb{R}$ by $f_{a_2}(a_2) = b_2$. Keep excising the chosen points from $\mathbb{N}$ and $\mathbb{R}$, and continue defining similar functions f_a for all $a \in \mathbb{N}$. Now, in exactly the same manner, define $f_{b_1} \colon \mathbb{R} \to \mathbb{N}$ by $f_{b_1}(b_1) = a_1$, and define such functions f_b for all $b \in \mathbb{R}$. What we've done is connect $\mathbb{N}$ to $\mathbb{R}$ with "strands" or arrows mapping one set into the other pointwise, and then showing that each arrowhead can be reversed. It remains to bunch up the strands into a single function, so to that end we define

$$f^{\mathbb{N}} := \bigcup_{a \in \mathbb{N}} f_a; \qquad f^{\mathbb{R}} := \bigcup_{b \in \mathbb{R}} f_b. \tag{4.6}$$

Each f_a corresponds to an f_b, so $f^{\mathbb{N}}$ and $f^{\mathbb{R}}$ are inverses and give a bijection $\mathbb{N} \to \mathbb{R}$. ■

4.3 Epilogue

We close with a meta-theorem on proofs:

Theorem 4.3. *There are infinitely many proofs that* $1 = 2$*.*

Proof. We proceed by induction. Clearly $1 \neq 2$, so there are zero proofs that $1 = 2$. (Alternatively, one could argue that the case $n = 1$ has been amply substantiated by any of the preceding proofs 1–10.) Assuming that there are k proofs, there are immediately $k + 1 - 1 = k + 2 - 1 = k + 1$ proofs that $1 = 2$, so the induction is complete and one finds that there are arbitrarily many proofs of this deep and subtle truth. ■